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A STRATEGIC VIEW ON THE CONSEQUENCES OF CLASSICAL INTEGRAL SUB-STRIPS AND COUPLED NONLOCAL MULTI-POINT BOUNDARY CONDITIONS ON A COMBINED CAPUTO FRACTIONAL DIFFERENTIAL EQUATION

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ABSTRACT. A versatile and unique approach has been handled with a combined Caputo differential equation with integral sub-strip boundary conditions and also with ally non local conditions. For the validity and applicability of the above problem we have proposed a convergent and stable method. Sufficient example has also been supplemented to substantiate the proof.

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1. Introduction

Ever since the evolution of Differential equations of fractional order there have been intense efforts to raise the theoretical and application aspects in various fields like environmental issues, classical mechanics, electron chemistry, aero dynamics, biological sciences, etc. In recent years, our researchers have indeed given numerous contributions in these areas. For details we refer the reader to the papers [7]-[11] and references therein. The boundary value problems has been established in recent years with a strong connection to the development of classical calculus. Moreover, some analytical results and applications of fractional calculus have been outlined in their historical context, for instance, see [1]-[3],[15]-[17] and the references cited therein. Recently, much interest has been created in establishing the existence of solutions for coupled system of fractional order boundary value problem with multi-point, integral-strip, sub-strips conditions. Tariboon et.al [5] experimented the coupled systems of Riemann-Liouville fractional differential equations with Hadamard fractional integral boundary conditions. Wang et.al [6] investigated on analysis of fractional order differential coupled systems. Agarwal et.al [13] experimented the new kind of nonlocal coupled flux and multi-point boundary value problem of coupled nonlinear fractional differential equations. Ahmad et.al. [4],[12],[14], profound the idea of new kind of nonlocal multi-point boundary value problem of fractional integro-differential equations involving multi-point strips integral boundary conditions.

In this paper the existence and uniqueness of solutions for the below coupled Caputo fractional differential equations of order ξ, ζ with coupled nonlocal multi-point Riemann-Liouville integral substrips boundary conditions are discussed. Consider the fractional differential equations

$$(1)\qquad \begin{cases} {}^c\mathfrak{D}^{\xi}p(z) = h_1(z, p(z), q(z), {}^c\mathfrak{D}^{\delta}q(z)), \\ {}^c\mathfrak{D}^{\zeta}q(z) = h_2(z, p(z), {}^c\mathfrak{D}^{\gamma}p(z), q(z)), \\ z \in \mathfrak{J} = [0,1],\ n-1 < \xi, \zeta \le n,\ n > 2,\ 0 < \delta, \gamma < 1 \end{cases}$$

$$(2)\qquad \begin{cases} p(0) = \varphi_1(q),\ p'(0) = \rho_1\int_{\nu_1}^{\omega_1} q'(\sigma)d\sigma,\ p''(0) = 0,\ p'''(0) = 0, \\ \cdots, p^{n-2}(0) = 0,\ p'(1) = \alpha_1\int_{\nu_2}^{\omega_2} q'(\sigma)d\sigma + \beta_1\sum_{j=1}^{k-2}\upsilon_j q'(\varsigma_j), \\ q(0) = \varphi_2(p),\ q'(0) = \rho_2\int_{\nu_1}^{\omega_1} p'(\sigma)d\sigma,\ q''(0) = 0,\ q'''(0) = 0, \\ \cdots, q^{n-2}(0) = 0,\ q'(1) = \alpha_2\int_{\nu_2}^{\omega_2} p'(\sigma)d\sigma + \beta_2\sum_{j=1}^{k-2}\upsilon_j p'(\varsigma_j), \end{cases}$$

where ${}^c\mathfrak{D}^{\xi}, {}^c\mathfrak{D}^{\zeta}, {}^c\mathfrak{D}^{\delta}, {}^c\mathfrak{D}^{\gamma}$ denote the Caputo fractional derivatives and h_1, h_2: $\mathfrak{J} \times \mathbb{R}^3$ to $\mathbb{R}$ and φ_1, φ_2: $C(\mathfrak{J}, \mathbb{R})$ to $\mathbb{R}$, are given continuous functions, $0 < \nu_1 < \omega_1 < \nu_2 < \omega_2 < \varsigma_1 < \varsigma_2 < \cdots < \varsigma_{k-2} < 1$, υ_j, $j = 1, 2, \cdots, k-2$, $\rho_j, \alpha_j, \beta_j$, $j = 1, 2$ are positive real constants. The rest of the paper is organised as follows: The preliminaries section is devoted to some fundamental concepts of fractional calculus with basic lemma related to the given problem. In section 3, the existence and uniqueness of solutions are obtained based on Leray-Schauder alternative, Banach fixed point theorem and also the validation of the results is done by providing examples.

2. Preliminaries

In this section, we introduce some notations and definitions of fractional calculus and present preliminary results needed in our proof later [1], [16].

Definition 2.1 The fractional integral of order ξ with the lower limit zero for a function h is defined as

$$\mathfrak{J}^{\xi}h(z) = \frac{1}{\Gamma(\xi)}\int_0^z \frac{h(\sigma)}{(z-\sigma)^{1-\xi}}d\sigma, \quad z > 0,\ \xi > 0,$$

provided the right hand-side is point-wise defined on $[0, \infty)$, where $\Gamma(\cdot)$ is the gamma function, which is defined by $\Gamma(\xi) = \int_0^{\infty} z^{\xi-1}e^{-z}dz$.

Definition 2.2 The Riemann-Liouville fractional derivative of order $\xi > 0$, $n-1 < \xi < n$, $n \in \mathbb{N}$ is defined as

$$\mathfrak{D}^{\xi}_{0+}h(z) = \frac{1}{\Gamma(n-\xi)}\left(\frac{d}{dz}\right)^n \int_0^z (z-\sigma)^{n-\xi-1}h(\sigma)d\sigma,$$

where the function $h(z)$ has absolutely continuous derivative up to order $(n-1)$.

Definition 2.3 The Caputo derivative of order ξ for a function $h : [0, \infty) \to \mathbb{R}$ can be written as

$${}^c\mathfrak{D}^{\xi}h(z) = \mathfrak{D}^{\xi}\left(h(z) - \sum_{j=0}^{n-1}\frac{z^j}{j!}h^{(j)}(0)\right), \ z > 0,\ n-1 < \xi < n.$$

Lemma 2.1. *For $\hat{h}_1, \hat{h}_2 \in C(\mathfrak{J})$, $p, q \in C^n(\mathfrak{J}, \mathbb{R})$, the solution of the linear system of fractional differential equations*

$$^{c}\mathfrak{D}^{\xi} p(z) = \hat{h}_1(z),\ {}^{c}\mathfrak{D}^{\zeta} q(z) = \hat{h}_2(z),\ n-1 < \xi, \zeta \leq n,\ n > 2,\ z \in \mathfrak{J}, \tag{3}$$

subject to the boundary conditions (2) is equivalent to the system of fractional integral equation

$$\begin{cases} p(z) \ = \ \displaystyle\int_0^z \frac{(z-\sigma)^{\xi-1}}{\Gamma(\xi)} \hat{h}_1(\sigma) d\sigma + \varphi_1(q) \\ +\Phi_1(z)\rho_1 \displaystyle\int_{\nu_1}^{\omega_1} \left(\int_0^{\sigma} \frac{(\sigma-\theta)^{\zeta-2}}{\Gamma(\zeta-1)} \hat{h}_2(\theta) d\theta \right) d\sigma \\ +\Phi_2(z)\rho_2 \displaystyle\int_{\nu_1}^{\omega_1} \left(\int_0^{\sigma} \frac{(\sigma-\theta)^{\xi-2}}{\Gamma(\xi-1)} \hat{h}_1(\theta) d\theta \right) d\sigma \\ +\Phi_3(z) \Bigg[\alpha_1 \displaystyle\int_{\nu_2}^{\omega_2} \left(\int_0^{\sigma} \frac{(\sigma-\theta)^{\zeta-2}}{\Gamma(\zeta-1)} \hat{h}_2(\theta) d\theta \right) d\sigma \\ +\beta_1 \displaystyle\sum_{j=1}^{k-2} \upsilon_j \int_0^{\varsigma_j} \frac{(\varsigma_j-\sigma)^{\zeta-2}}{\Gamma(\zeta-1)} \hat{h}_2(\sigma) d\sigma - \int_0^1 \frac{(1-\sigma)^{\xi-2}}{\Gamma(\xi-1)} \hat{h}_1(\sigma) d\sigma \Bigg] \\ +\Phi_4(z) \Bigg[\alpha_2 \displaystyle\int_{\nu_2}^{\omega_2} \left(\int_0^{\sigma} \frac{(\sigma-\theta)^{\xi-2}}{\Gamma(\xi-1)} \hat{h}_1(\theta) d\theta \right) d\sigma \\ +\beta_2 \displaystyle\sum_{j=1}^{k-2} \upsilon_j \int_0^{\varsigma_j} \frac{(\varsigma_j-\sigma)^{\xi-2}}{\Gamma(\xi-1)} \hat{h}_1(\sigma) d\sigma - \int_0^1 \frac{(1-\sigma)^{\zeta-2}}{\Gamma(\zeta-1)} \hat{h}_2(\sigma) d\sigma \Bigg], \end{cases} \tag{4}$$

$$\begin{cases} q(z) = \displaystyle\int_0^z \frac{(z-\sigma)^{\zeta-1}}{\Gamma(\zeta)} \hat{h}_2(\sigma) d\sigma + \varphi_2(p) \\ +\Phi_5(z)\rho_1 \displaystyle\int_{\nu_1}^{\omega_1} \left(\int_0^{\sigma} \frac{(\sigma-\theta)^{\zeta-2}}{\Gamma(\zeta-1)} \hat{h}_2(\theta) d\theta \right) d\sigma \\ +\Phi_6(z)\rho_2 \displaystyle\int_{\nu_1}^{\omega_1} \left(\int_0^{\sigma} \frac{(\sigma-\theta)^{\xi-2}}{\Gamma(\xi-1)} \hat{h}_1(\theta) d\theta \right) d\sigma \\ +\Phi_7(z) \Bigg[\alpha_1 \displaystyle\int_{\nu_2}^{\omega_2} \left(\int_0^{\sigma} \frac{(\sigma-\theta)^{\zeta-2}}{\Gamma(\zeta-1)} \hat{h}_2(\theta) d\theta \right) d\sigma \\ +\beta_1 \displaystyle\sum_{j=1}^{k-2} \upsilon_j \int_0^{\varsigma_j} \frac{(\varsigma_j-\sigma)^{\zeta-2}}{\Gamma(\zeta-1)} \hat{h}_2(\sigma) d\sigma - \int_0^1 \frac{(1-\sigma)^{\xi-2}}{\Gamma(\xi-1)} \hat{h}_1(\sigma) d\sigma \Bigg] \\ +\Phi_8(z) \Bigg[\alpha_2 \displaystyle\int_{\nu_2}^{\omega_2} \left(\int_0^{\sigma} \frac{(\sigma-\theta)^{\xi-2}}{\Gamma(\xi-1)} \hat{h}_1(\theta) d\theta \right) d\sigma \\ +\beta_2 \displaystyle\sum_{j=1}^{k-2} \upsilon_j \int_0^{\varsigma_j} \frac{(\varsigma_j-\sigma)^{\xi-2}}{\Gamma(\xi-1)} \hat{h}_1(\sigma) d\sigma - \int_0^1 \frac{(1-\sigma)^{\zeta-2}}{\Gamma(\zeta-1)} \hat{h}_2(\sigma) d\sigma \Bigg], \end{cases} \tag{5}$$

where

$$(6)\quad \begin{cases} \Phi_1(z)=\frac{z\vartheta_1}{\lambda_1}+\frac{z^{n-1}\sigma_1}{\Delta}, \Phi_2(z)=\frac{z\vartheta_2}{\lambda_1}+\frac{z^{n-1}\sigma_2}{\Delta}, \Phi_3(z)=\frac{z\vartheta_3}{\lambda_1}-\frac{z^{n-1}\sigma_3}{\Delta},\\ \Phi_4(z)=\frac{z\vartheta_4}{\lambda_1}-\frac{z^{n-1}\sigma_4}{\Delta}, \Phi_5(z)=\frac{z\vartheta_5}{\lambda_1}-\frac{z^{n-1}\sigma_5}{\Delta}, \Phi_6(z)=\frac{z\vartheta_6}{\lambda_1}-\frac{z^{n-1}\sigma_6}{\Delta},\\ \Phi_7(z)=\frac{z\vartheta_7}{\lambda_1}+\frac{z^{n-1}\sigma_7}{\Delta}, \Phi_8(z)=\frac{z\vartheta_8}{\lambda_1}+\frac{z^{n-1}\sigma_8}{\Delta}, \end{cases}$$

$$(7)\quad \begin{cases} \vartheta_1=1+\frac{\varpi_1\varpi_4\sigma_1-\varpi_2\sigma_5}{\Delta}, \vartheta_4=\frac{\varpi_2\sigma_8-\varpi_1\varpi_4\sigma_4}{\Delta}, \vartheta_7=\frac{\varpi_2\varpi_3\sigma_7-\varpi_4\sigma_3}{\Delta},\\ \vartheta_2=\varpi_1+\frac{\varpi_1\varpi_4\sigma_2-\varpi_2\sigma_6}{\Delta}, \vartheta_5=\varpi_3+\frac{\varpi_4\sigma_1-\varpi_2\varpi_3\sigma_5}{\Delta},\\ \vartheta_3=\frac{\varpi_2\sigma_7-\varpi_1\varpi_4\sigma_3}{\Delta}, \vartheta_6=1+\frac{\varpi_4\sigma_2-\varpi_2\varpi_3\sigma_6}{\Delta}, \vartheta_8=\frac{\varpi_2\varpi_3\sigma_8-\varpi_4\sigma_4}{\Delta}, \end{cases}$$

$$(8)\quad \begin{cases} \sigma_1=\lambda_2(\Delta_4-\Delta_3\varpi_3),\ \sigma_2=\lambda_2(\Delta_4\varpi_1-\Delta_3),\ \sigma_3=\lambda_1(\Delta_4-\Delta_3\eta_3),\\ \sigma_4=\lambda_1(\Delta_4\eta_1-\Delta_3),\ \sigma_5=\lambda_2(\Delta_2-\Delta_1\varpi_3),\ \sigma_6=\lambda_2(\Delta_2\varpi_1-\Delta_1),\\ \sigma_7=\lambda_1(\Delta_2-\Delta_1\eta_3),\ \sigma_8=\lambda_1(\Delta_2\eta_1-\Delta_1), \end{cases}$$

$$(9)\quad \begin{cases} \Delta_1=\varpi_1\varpi_4\lambda_2+\lambda_1((n-1)-\eta_1\eta_4), \Delta_2=\varpi_4\lambda_2+\lambda_1((n-1)\eta_3-\eta_4),\\ \Delta_3=\varpi_2\lambda_2+\lambda_1((n-1)\eta_1-\eta_2), \Delta_4=\varpi_2\varpi_3\lambda_2+\lambda_1((n-1)-\eta_2\eta_3), \end{cases}$$

$$(10)\quad \begin{cases} \varpi_1 = \rho_1(\omega_1-\nu_1), & \varpi_2 = \rho_1(\omega_1^{n-1}-\nu_1^{n-1}),\\ \varpi_3 = \rho_2(\omega_1-\nu_1), & \varpi_4 = \rho_2(\omega_1^{n-1}-\nu_1^{n-1}), \end{cases}$$

$$(11)\quad \begin{cases} \eta_1=\alpha_1(\omega_2-\nu_2)+\beta_1\sum_{j=1}^{k-2}\upsilon_j,\ \eta_2=\alpha_1(\omega_2^{n-1}-\nu_2^{n-1})+(n-1)\beta_1\sum_{j=1}^{k-2}\upsilon_j\varsigma_j^{n-2},\\ \eta_3=\alpha_2(\omega_2-\nu_2)+\beta_2\sum_{j=1}^{k-2}\upsilon_j,\ \eta_4=\alpha_2(\omega_2^{n-1}-\nu_2^{n-1})+(n-1)\beta_2\sum_{j=1}^{k-2}\upsilon_j\varsigma_j^{n-2}, \end{cases}$$

$$(12)\quad \{\ \Delta=\Delta_2\Delta_3-\Delta_1\Delta_4\neq 0,\ \lambda_1 = 1-\varpi_1\varpi_3,\ \lambda_2 = 1-\eta_1\eta_3,$$

3. Existence and Uniqueness Results

In this section, we shall obtain the existence and uniqueness results of the BVP (1) and (2). For $0<\gamma,\delta<1$, We define spaces $\mathfrak{P}=\{p : p\in C(\mathfrak{J},\mathbb{R}), {}^c\mathfrak{D}^{\gamma}p\in C(\mathfrak{J},\mathbb{R})\}$ and $\mathfrak{Q}=\{q : q\in C(\mathfrak{J},\mathbb{R}), {}^c\mathfrak{D}^{\delta}q\in C(\mathfrak{J},\mathbb{R})\}$ equipped respectively with norms $\|p\|_{\mathfrak{P}}=\|p\|+\|{}^c\mathfrak{D}^{\gamma}p\|=\sup_{z\in\mathfrak{J}}|p(z)|+\sup_{z\in\mathfrak{J}}|{}^c\mathfrak{D}^{\gamma}p(z)|$, and $\|q\|_{\mathfrak{Q}}=\|q\|+\|{}^c\mathfrak{D}^{\delta}q\|=\sup_{z\in\mathfrak{J}}|q(z)|+\sup_{z\in\mathfrak{J}}|{}^c\mathfrak{D}^{\delta}q(z)|$. Therefore, $(\mathfrak{P},\|\cdot\|_{\mathfrak{P}})$ and $(\mathfrak{Q},\|\cdot\|_{\mathfrak{Q}})$ are Banach spaces and consequently, the product space $(\mathfrak{P}\times\mathfrak{Q},\|\cdot\|_{\mathfrak{P}\times\mathfrak{Q}})$ is a Banach space with norm $\|(p,q)\|_{\mathfrak{P}\times\mathfrak{Q}}=\|p\|_{\mathfrak{P}}+\|q\|_{\mathfrak{Q}}$ for $(p,q)\in\mathfrak{P}\times\mathfrak{Q}$. In view of Lemma 2.1, we define an operator $\mathfrak{T}:\mathfrak{P}\times\mathfrak{Q}\to\mathfrak{P}\times\mathfrak{Q}$ as

$$(13)\qquad \mathfrak{T}(p,q)(z) = (\mathfrak{T}_1(p,q)(z),\mathfrak{T}_2(p,q)(z))$$

where

$$\mathfrak{T}_1(p,q)(z) = \int_0^z \frac{(z-\sigma)^{\xi-1}}{\Gamma(\xi)}h_1(\sigma,p(\sigma),q(\sigma),{}^c\mathfrak{D}^{\delta}q(\sigma))d\sigma+\varphi_1(q)$$

$$
\begin{aligned}
& +\Phi_1(z)\left[\rho_1\int_{\nu_1}^{\omega_1}\left(\int_0^{\sigma}\frac{(\sigma-\theta)^{\zeta-2}}{\Gamma(\zeta-1)}h_2(\theta,p(\theta),{}^c\mathfrak{D}^{\gamma}p(\theta),q(\theta))d\theta\right)d\sigma\right]\\
& +\Phi_2(z)\left[\rho_2\int_{\nu_1}^{\omega_1}\left(\int_0^{\sigma}\frac{(\sigma-\theta)^{\xi-2}}{\Gamma(\xi-1)}h_1(\theta,p(\theta),q(\theta),{}^c\mathfrak{D}^{\delta}q(\theta))d\theta\right)d\sigma\right]\\
& +\Phi_3(z)\left[\alpha_1\int_{\nu_2}^{\omega_2}\left(\int_0^{\sigma}\frac{(\sigma-\theta)^{\zeta-2}}{\Gamma(\zeta-1)}h_2(\theta,p(\theta),{}^c\mathfrak{D}^{\gamma}p(\theta),q(\theta))d\theta\right)d\sigma\right.\\
& +\beta_1\sum_{j=1}^{k-2}\upsilon_j\int_0^{\varsigma_j}\frac{(\varsigma_j-\sigma)^{\zeta-2}}{\Gamma(\zeta-1)}h_2(\sigma,p(\sigma),{}^c\mathfrak{D}^{\gamma}p(\sigma),q(\sigma))d\sigma\\
& \left.-\int_0^1\frac{(1-\sigma)^{\xi-2}}{\Gamma(\xi-1)}h_1(\sigma,p(\sigma),q(\sigma),{}^c\mathfrak{D}^{\delta}q(\sigma))d\sigma\right]\\
& +\Phi_4(z)\left[\alpha_2\int_{\nu_2}^{\omega_2}\left(\int_0^{\sigma}\frac{(\sigma-\theta)^{\xi-2}}{\Gamma(\xi-1)}h_1(\theta,p(\theta),q(\theta),{}^c\mathfrak{D}^{\delta}q(\theta))d\theta\right)d\sigma\right.\\
& +\beta_2\sum_{j=1}^{k-2}\upsilon_j\int_0^{\varsigma_j}\frac{(\varsigma_j-\sigma)^{\xi-2}}{\Gamma(\xi-1)}h_1(\sigma,p(\sigma),q(\sigma),{}^c\mathfrak{D}^{\delta}q(\sigma))d\sigma\\
& \left.-\int_0^1\frac{(1-\sigma)^{\zeta-2}}{\Gamma(\zeta-1)}h_2(\sigma,p(\sigma),{}^c\mathfrak{D}^{\gamma}p(\sigma),q(\sigma))d\sigma\right] \qquad (14)\\
\mathfrak{T}_2(p,q)(z) \;=\; & \int_0^z\frac{(z-\sigma)^{\zeta-1}}{\Gamma(\zeta)}h_2(\sigma,p(\sigma),{}^c\mathfrak{D}^{\gamma}p(\sigma),q(\sigma))d\sigma+\varphi_2(p)\\
& +\Phi_5(z)\left[\rho_1\int_{\nu_1}^{\omega_1}\left(\int_0^{\sigma}\frac{(\sigma-\theta)^{\zeta-2}}{\Gamma(\zeta-1)}h_2(\theta,p(\theta),{}^c\mathfrak{D}^{\gamma}p(\theta),q(\theta))d\theta\right)d\sigma\right]\\
& +\Phi_6(z)\left[\rho_2\int_{\nu_1}^{\omega_1}\left(\int_0^{\sigma}\frac{(\sigma-\theta)^{\xi-2}}{\Gamma(\xi-1)}h_1(\theta,p(\theta),q(\theta),{}^c\mathfrak{D}^{\delta}q(\theta))d\theta\right)d\sigma\right]\\
& +\Phi_7(z)\left[\alpha_1\int_{\nu_2}^{\omega_2}\left(\int_0^{\sigma}\frac{(\sigma-\theta)^{\zeta-2}}{\Gamma(\zeta-1)}h_2(\theta,p(\theta),{}^c\mathfrak{D}^{\gamma}p(\theta),q(\theta))d\theta\right)d\sigma\right.\\
& +\beta_1\sum_{j=1}^{k-2}\upsilon_j\int_0^{\varsigma_j}\frac{(\varsigma_j-\sigma)^{\zeta-2}}{\Gamma(\zeta-1)}h_2(\sigma,p(\sigma),{}^c\mathfrak{D}^{\gamma}p(\sigma),q(\sigma))d\sigma\\
& \left.-\int_0^1\frac{(1-\sigma)^{\xi-2}}{\Gamma(\xi-1)}h_1(\sigma,p(\sigma),q(\sigma),{}^c\mathfrak{D}^{\delta}q(\sigma))d\sigma\right]\\
& +\Phi_8(z)\left[\alpha_2\int_{\nu_2}^{\omega_2}\left(\int_0^{\sigma}\frac{(\sigma-\theta)^{\xi-2}}{\Gamma(\xi-1)}h_1(\theta,p(\theta),q(\theta),{}^c\mathfrak{D}^{\delta}q(\theta))d\theta\right)d\sigma\right.\\
& +\beta_2\sum_{j=1}^{k-2}\upsilon_j\int_0^{\varsigma_j}\frac{(\varsigma_j-\sigma)^{\xi-2}}{\Gamma(\xi-1)}h_1(\sigma,p(\sigma),q(\sigma),{}^c\mathfrak{D}^{\delta}q(\sigma))d\sigma
\end{aligned}
$$

$$-\int_0^1 \frac{(1-\sigma)^{\zeta-2}}{\Gamma(\zeta-1)} h_2(\sigma, p(\sigma), {}^c\mathfrak{D}^\gamma p(\sigma), q(\sigma))d\sigma\Bigg] \tag{15}$$

In the forthcoming analysis, we need the following assumptions:
($\mathfrak{H}_1$) The continuous functions h_1, h_2 are defined from $\mathfrak{J}\times\mathbb{R}^3$ to $\mathbb{R}$ and $\exists$ constants τ_i and $\varepsilon_i \geq 0$, $\tau_0 > 0, \varepsilon_0 > 0 \ni \forall\ w_i \in \mathbb{R},\ i = 1, 2, 3$, we have

$$\begin{aligned} |h_1(z, w_1, w_2, w_3)| &\leq \tau_0 + \tau_1|w_1| + \tau_2|w_2| + \tau_3|w_3|,\\ |h_2(z, w_1, w_2, w_3)| &\leq \varepsilon_0 + \varepsilon_1|w_1| + \varepsilon_2|w_2| + \varepsilon_3|w_3|. \end{aligned}$$

($\mathfrak{H}_2$) The continuous functions φ_1, φ_2 are defined from $C(\mathfrak{J}, \mathbb{R})$ to $\mathbb{R}$ with $\varphi_1 = \varphi_2 = 0$ and $\exists$ constants $\mu_i,\ i = 1, 2$, we have

$$\begin{aligned} |\varphi_1(q)| &\leq \mu_1|q|,\\ |\varphi_2(p)| &\leq \mu_2|p|. \end{aligned}$$

($\mathfrak{H}_3$) The continuous functions h_1, h_2 are defined from $\mathfrak{J}\times\mathbb{R}^3$ to $\mathbb{R}$ and $\exists$ positive constants $\mathfrak{G}_1$ and $\mathfrak{G}_2 \ni \forall z \in \mathfrak{J}$ and $w_i, v_i \in \mathbb{R}(i = 1, 2, 3)$, we have

$$\begin{aligned} |h_1(z, w_1, w_2, w_3) - h_1(z, v_1, v_2, v_3)| &\leq \mathfrak{G}_1(|w_1 - v_1| + |w_2 - v_2| + |w_3 - v_3|),\\ |h_2(z, w_1, w_2, w_3) - h_2(z, v_1, v_2, v_3)| &\leq \mathfrak{G}_2(|w_1 - v_1| + |w_2 - v_2| + |w_3 - v_3|). \end{aligned}$$

($\mathfrak{H}_4$) The continuous functions φ_1, φ_2 are defined from $C(\mathfrak{J}, \mathbb{R})$ to $\mathbb{R}$ with $\varphi_1 = \varphi_2 = 0$ and $\exists$ constants ϱ_i, $\ni \forall\ w_i \in C(\mathfrak{J}, \mathbb{R}),\ i = 1, 2$, we have

$$\begin{aligned} |\varphi_1(w_1) - \varphi_1(w_2)| &\leq \varrho_1|w_1 - w_2|,\\ |\varphi_2(w_1) - \varphi_2(w_2)| &\leq \varrho_2|w_1 - w_2|. \end{aligned}$$

To avoid computational complexity, we assume

$$\Omega_1 = \frac{1}{\Gamma(\xi+1)}\Big[1+\widehat{\Phi}_2(\rho_2(\omega_1^\xi-\nu_1^\xi))+\xi\widehat{\Phi}_3+\widehat{\Phi}_4[\alpha_2(\omega_2^\xi-\nu_2^\xi)+\xi\beta_2\sum_{j=1}^{k-2}\upsilon_j\varsigma_j^{\xi-1}]\Big],$$

$$\hat{\Omega}_1 = \frac{1}{\Gamma(\zeta+1)}\Big[\widehat{\Phi}_1(\rho_1(\omega_1^\zeta - \nu_1^\zeta)) + \zeta\widehat{\Phi}_4 + \widehat{\Phi}_3[\alpha_1(\omega_2^\zeta - \nu_2^\zeta) + \zeta\beta_1\sum_{j=1}^{k-2}\upsilon_j\varsigma_j^{\zeta-1}]\Big],$$

$$\Omega_2 = \frac{1}{\Gamma(\xi)}\left[1 + \frac{\widehat{\Phi}'_2(\rho_2(\omega_1^\xi - \nu_1^\xi))}{\xi} + \widehat{\Phi}'_3 + \widehat{\Phi}'_4\Big[\frac{\alpha_2(\omega_2^\xi - \nu_2^\xi)}{\xi} + \beta_2\sum_{j=1}^{k-2}\upsilon_j\varsigma_j^{\xi-1}\Big]\right],$$

$$\hat{\Omega}_2 = \frac{1}{\Gamma(\zeta+1)}\Big[\widehat{\Phi}'_1(\rho_1(\omega_1^\zeta - \nu_1^\zeta)) + \zeta\widehat{\Phi}'_4 + \widehat{\Phi}'_3[\alpha_1(\omega_2^\zeta - \nu_2^\zeta) + \zeta\beta_1\sum_{j=1}^{k-2}\upsilon_j\varsigma_j^{\zeta-1}]\Big],$$

$$\Omega_3 = \frac{1}{\Gamma(\zeta+1)}\Big[1+\widehat{\Phi}_5(\rho_1(\omega_1^\zeta-\nu_1^\zeta))+\zeta\widehat{\Phi}_8+\widehat{\Phi}_7[\alpha_1(\omega_2^\zeta-\nu_2^\zeta)+\zeta\beta_1\sum_{j=1}^{k-2}\upsilon_j\varsigma_j^{\zeta-1}]\Big],$$

$$\hat{\Omega}_3 = \frac{1}{\Gamma(\xi+1)}\Big[\widehat{\Phi}_6(\rho_2(\omega_1^\xi - \nu_1^\xi)) + \xi\widehat{\Phi}_7 + \widehat{\Phi}_8[\alpha_2(\omega_2^\xi - \nu_2^\xi) + \xi\beta_2\sum_{j=1}^{k-2}\upsilon_j\varsigma_j^{\xi-1}]\Big],$$

$$\Omega_4 = \frac{1}{\Gamma(\zeta)}\left[1 + \frac{\widehat{\Phi}'_5(\rho_1(\omega_1^\zeta - \nu_1^\zeta))}{\zeta} + \widehat{\Phi}'_8 + \widehat{\Phi}'_7\Big[\frac{\alpha_1(\omega_2^\zeta - \nu_2^\zeta)}{\zeta} + \beta_1\sum_{j=1}^{k-2}\upsilon_j\varsigma_j^{\zeta-1}\Big]\right],$$

$$\hat{\Omega}_4 = \frac{1}{\Gamma(\xi{+}1)}\Big[\widehat{\Phi}_6'(\rho_2(\omega_1^{\xi}-\nu_1^{\xi}))+\xi\widehat{\Phi}_7'+\widehat{\Phi}_8'[\alpha_2(\omega_2^{\xi}-\nu_2^{\xi})+\xi\beta_2\sum_{j=1}^{k-2}\upsilon_j\varsigma_j^{\xi-1}]\Big],$$

$$\widehat{\Phi}_j=\max_{z\in\mathfrak{J}}|\Phi_j(z)| \text{ and } \widehat{\Phi}_j'=\max_{z\in\mathfrak{J}}|\Phi_j'(z)|,\ j=1,2,...,8.$$

Theorem 3.1. *Let $\mathfrak{T}:\mathfrak{F}\to\mathfrak{F}$ be a completely continuous operator (i.e., a map that restricted to any bounded set in $\mathfrak{F}$ is compact). Let $\chi(\mathfrak{T})=\{p\in\mathfrak{E}: p=\epsilon\mathfrak{T}(p)$ for some $0<\epsilon<1\}$. Then either the set $\chi(\mathfrak{T})$ is unbounded, or $\mathfrak{T}$ has at least one fixed point.*

To run the interference for the proof, we introduce the notations.

$$\mathfrak{V}_1 = \Omega_1\tau_0+\hat{\Omega}_1\varepsilon_0+\frac{\Big[\Omega_2\tau_0+\hat{\Omega}_2\varepsilon_0\Big]}{\Gamma(2-\gamma)}+\Omega_3\varepsilon_0+\hat{\Omega}_3\tau_0+\frac{\Big[\Omega_4\varepsilon_0+\hat{\Omega}_4\tau_0\Big]}{\Gamma(2-\delta)}, \tag{16}$$

$$\begin{aligned}\mathfrak{V}_2 &= \Omega_1\tau_1+\hat{\Omega}_1 max\{\varepsilon_1,\varepsilon_2\}+\mu_2+\frac{1}{\Gamma(2-\gamma)}\Big[\Omega_2\tau_1+\hat{\Omega}_2 max\{\varepsilon_1,\varepsilon_2\}\Big]\\ &\quad+\Omega_3 max\{\varepsilon_1,\varepsilon_2\}+\hat{\Omega}_3\tau_1+\frac{1}{\Gamma(2-\delta)}\Big[\Omega_4 max\{\varepsilon_1,\varepsilon_2\}+\hat{\Omega}_4\tau_1\Big],\end{aligned} \tag{17}$$

$$\begin{aligned}\mathfrak{V}_3 &= \Omega_1 max\{\tau_2,\tau_3\}+\hat{\Omega}_1\varepsilon_3+\mu_1+\frac{1}{\Gamma(2-\gamma)}\Big[\Omega_2 max\{\tau_2,\tau_3\}+\hat{\Omega}_2\varepsilon_3\Big]\\ &\quad+\Omega_3\varepsilon_3+\hat{\Omega}_3 max\{\tau_2,\tau_3\}+\frac{1}{\Gamma(2-\delta)}\Big[\Omega_4\varepsilon_3+\hat{\Omega}_4 max\{\tau_2,\tau_3\}\Big],\end{aligned} \tag{18}$$

$$\hat{\mathfrak{V}} = max\{\mathfrak{V}_2,\mathfrak{V}_3\}. \tag{19}$$

Theorem 3.2. *Let us speculate that $(\mathfrak{H}_1)$ and $(\mathfrak{H}_2)$ holds. Then $\exists$ at least one solution for problem (1) and (2) on $\mathfrak{J}$ if $\hat{\mathfrak{V}}<1$, where $\hat{\mathfrak{V}}$ is given by (19).*

Proof. In the first step, we show that the operator $\mathfrak{T}:\mathfrak{P}\times\mathfrak{Q}\to\mathfrak{P}\times\mathfrak{Q}$ is completely continuous. By continuity of the functions h_1,h_2, it follows that the operator $\mathfrak{T}$ is continuous. $\Psi\subset\mathfrak{P}\times\mathfrak{Q}$ be bounded. Then $\exists$ positive constants $\mathfrak{M}_{h_1}$, $\mathfrak{M}_{h_2}$ such that

$$|h_1(z,p(z),q(z),{}^c\mathfrak{D}^{\delta}q(z))|\le\mathfrak{M}_{h_1},\quad |h_2(z,p(z),{}^c\mathfrak{D}^{\gamma}p(z),q(z))|\le\mathfrak{M}_{h_2},$$

$\forall\ (p,q)\in\Psi$, and constants $\mathfrak{M}_{\varphi_1},\mathfrak{M}_{\varphi_2}>0\ni|\varphi_1(q)|\le\mathfrak{M}_{\varphi_1}$,
$|\varphi_2(p)|\le\mathfrak{M}_{\varphi_2}$, $\forall\ (p,q)\in C(\mathfrak{J},\mathbb{R})$. So, for any $(p,q)\in\Psi$, we have

$$\begin{aligned}|\mathfrak{T}_1(p,q)(z)| \le\ & \int_0^z\frac{(z-\sigma)^{\xi-1}}{\Gamma(\xi)}\mathfrak{M}_{h_1}d\sigma+\mathfrak{M}_{\varphi_1}\\ &+\Phi_1(z)\left[\rho_1\int_{\nu_1}^{\omega_1}\left(\int_0^{\sigma}\frac{(\sigma-\theta)^{\zeta-2}}{\Gamma(\zeta-1)}\mathfrak{M}_{h_2}d\theta\right)d\sigma\right]\\ &+\Phi_2(z)\left[\rho_2\int_{\nu_1}^{\omega_1}\left(\int_0^{\sigma}\frac{(\sigma-\theta)^{\xi-2}}{\Gamma(\xi-1)}\mathfrak{M}_{h_1}d\theta\right)d\sigma\right]\\ &+\Phi_3(z)\left[\alpha_1\int_{\nu_2}^{\omega_2}\left(\int_0^{\sigma}\frac{(\sigma-\theta)^{\zeta-2}}{\Gamma(\zeta-1)}\mathfrak{M}_{h_2}d\theta\right)d\sigma\right.\\ &\left.+\beta_1\sum_{j=1}^{k-2}\upsilon_j\int_0^{\varsigma_j}\frac{(\varsigma_j-\sigma)^{\zeta-2}}{\Gamma(\zeta-1)}\mathfrak{M}_{h_2}d\sigma+\int_0^1\frac{(1-\sigma)^{\xi-2}}{\Gamma(\xi-1)}\mathfrak{M}_{h_1}d\sigma\right]\end{aligned}$$

$$
\begin{aligned}
& +\Phi_4(z)\Bigg[\alpha_2\int_{\nu_2}^{\omega_2}\left(\int_0^{\sigma}\frac{(\sigma-\theta)^{\xi-2}}{\Gamma(\xi-1)}\mathfrak{M}_{h_1}d\theta\right)d\sigma \\
& +\beta_2\sum_{j=1}^{k-2}\upsilon_j\int_0^{\varsigma_j}\frac{(\varsigma_j-\sigma)^{\xi-2}}{\Gamma(\xi-1)}\mathfrak{M}_{h_1}d\sigma+\int_0^1\frac{(1-\sigma)^{\zeta-2}}{\Gamma(\zeta-1)}\mathfrak{M}_{h_2}d\sigma\Bigg] \\
\leq\ & \mathfrak{M}_{h_1}\Omega_1+\mathfrak{M}_{h_2}\hat{\Omega}_1+\mathfrak{M}_{\varphi_1},
\end{aligned}
$$

$$
\begin{aligned}
|\mathfrak{T}_1'(p,q)(z)| \leq\ & \int_0^z\frac{(z-\sigma)^{\xi-2}}{\Gamma(\xi-1)}\mathfrak{M}_{h_1}d\sigma \\
& +\Phi_1'(z)\Bigg[\rho_1\int_{\nu_1}^{\omega_1}\left(\int_0^{\sigma}\frac{(\sigma-\theta)^{\zeta-2}}{\Gamma(\zeta-1)}\mathfrak{M}_{h_2}d\theta\right)d\sigma\Bigg] \\
& +\Phi_2'(z)\Bigg[\rho_2\int_{\nu_1}^{\omega_1}\left(\int_0^{\sigma}\frac{(\sigma-\theta)^{\xi-2}}{\Gamma(\xi-1)}\mathfrak{M}_{h_1}d\theta\right)d\sigma\Bigg] \\
& +\Phi_3'(z)\Bigg[\alpha_1\int_{\nu_2}^{\omega_2}\left(\int_0^{\sigma}\frac{(\sigma-\theta)^{\zeta-2}}{\Gamma(\zeta-1)}\mathfrak{M}_{h_2}d\theta\right)d\sigma \\
& +\beta_1\sum_{j=1}^{k-2}\upsilon_j\int_0^{\varsigma_j}\frac{(\varsigma_j-\sigma)^{\zeta-2}}{\Gamma(\zeta-1)}\mathfrak{M}_{h_2}d\sigma+\int_0^1\frac{(1-\sigma)^{\xi-2}}{\Gamma(\xi-1)}\mathfrak{M}_{h_1}d\sigma\Bigg] \\
& +\Phi_4'(z)\Bigg[\alpha_2\int_{\nu_2}^{\omega_2}\left(\int_0^{\sigma}\frac{(\sigma-\theta)^{\xi-2}}{\Gamma(\xi-1)}\mathfrak{M}_{h_1}d\theta\right)d\sigma \\
& +\beta_2\sum_{j=1}^{k-2}\upsilon_j\int_0^{\varsigma_j}\frac{(\varsigma_j-\sigma)^{\xi-2}}{\Gamma(\xi-1)}\mathfrak{M}_{h_1}d\sigma+\int_0^1\frac{(1-\sigma)^{\zeta-2}}{\Gamma(\zeta-1)}\mathfrak{M}_{h_2}d\sigma\Bigg] \\
\leq\ & \mathfrak{M}_{h_1}\Omega_2+\mathfrak{M}_{h_2}\hat{\Omega}_2,
\end{aligned}
$$

$$
\begin{aligned}
|{}^c\mathfrak{D}^{\gamma}\mathfrak{T}_1(p,q)(z)| &\leq \int_0^z\frac{(z-\sigma)^{-\gamma}}{\Gamma(1-\gamma)}|\mathfrak{T}_1'(p,q)(\sigma)|d\sigma \\
&\leq \frac{1}{\Gamma(2-\gamma)}\Big[\mathfrak{M}_{h_1}\Omega_2+\mathfrak{M}_{h_2}\hat{\Omega}_2\Big].
\end{aligned}
$$

Thus

$$
\begin{aligned}
\|\mathfrak{T}_1(p,q)\|_{\mathfrak{P}} &= \|\mathfrak{T}_1(p,q)\|+\|{}^c\mathfrak{D}^{\gamma}\mathfrak{T}_1(p,q)\| \\
&\leq \mathfrak{M}_{h_1}\Omega_1+\mathfrak{M}_{h_2}\hat{\Omega}_1+\mathfrak{M}_{\varphi_1}+\frac{1}{\Gamma(2-\gamma)}\Big[\mathfrak{M}_{h_1}\Omega_2+\mathfrak{M}_{h_2}\hat{\Omega}_2\Big]. \qquad (20)
\end{aligned}
$$

Similarly, we obtain

$$
\|\mathfrak{T}_2(p,q)\|_{\mathfrak{Q}} \leq \mathfrak{M}_{h_2}\Omega_3+\mathfrak{M}_{h_1}\hat{\Omega}_3+\mathfrak{M}_{\varphi_2}+\frac{1}{\Gamma(2-\delta)}\Big[\mathfrak{M}_{h_2}\Omega_4+\mathfrak{M}_{h_1}\hat{\Omega}_4\Big]. \qquad (21)
$$

Hence from (20) and (21), it follows that $\mathfrak{T}$ is uniformly bounded.

We shall proceed to prove that the operator $\mathfrak{T}$ is equicontinuous. For $z_1, z_2 \in \mathfrak{J}$ with $z_1 < z_2$, we have

$$
\begin{aligned}
&|\mathfrak{T}_1(p,q)(z_2)-\mathfrak{T}_1(p,q)(z_1)|\\
&\quad\le \left\{\left|\int_0^{z_1}\frac{[(z_2-\sigma)^{\xi-1}-(z_1-\sigma)^{\xi-1}]}{\Gamma(\xi)}d\sigma\right|+\left|\int_{z_1}^{z_2}\frac{(z_2-\sigma)^{\xi-1}}{\Gamma(\xi)}d\sigma\right|\right\}\mathfrak{M}_{h_1}\\
&\qquad+\frac{(z_2-z_1)\vartheta_1}{\lambda_1}+\frac{(z_2^{n-1}-z_1^{n-1})\sigma_1}{\Delta}\left[\rho_1\int_{\nu_1}^{\omega_1}\left(\int_0^{\sigma}\frac{(\sigma-\theta)^{\zeta-2}}{\Gamma(\zeta-1)}\mathfrak{M}_{h_2}d\theta\right)d\sigma\right]\\
&\qquad+\frac{(z_2-z_1)\vartheta_2}{\lambda_1}+\frac{(z_2^{n-1}-z_1^{n-1})\sigma_2}{\Delta}\left[\rho_2\int_{\nu_1}^{\omega_1}\left(\int_0^{\sigma}\frac{(\sigma-\theta)^{\xi-2}}{\Gamma(\xi-1)}\mathfrak{M}_{h_1}d\theta\right)d\sigma\right]\\
&\qquad+\frac{(z_2-z_1)\vartheta_3}{\lambda_1}+\frac{(z_2^{n-1}-z_1^{n-1})\sigma_3}{\Delta}\left[\alpha_1\int_{\nu_2}^{\omega_2}\left(\int_0^{\sigma}\frac{(\sigma-\theta)^{\zeta-2}}{\Gamma(\zeta-1)}\mathfrak{M}_{h_2}d\theta\right)d\sigma\right.\\
&\qquad\left.+\beta_1\sum_{j=1}^{k-2}\upsilon_j\int_0^{\varsigma_j}\frac{(\varsigma_j-\sigma)^{\zeta-2}}{\Gamma(\zeta-1)}\mathfrak{M}_{h_2}d\sigma+\int_0^1\frac{(1-\sigma)^{\xi-2}}{\Gamma(\xi-1)}\mathfrak{M}_{h_1}d\sigma\right]\\
&\qquad+\frac{(z_2-z_1)\vartheta_4}{\lambda_1}+\frac{(z_2^{n-1}-z_1^{n-1})\sigma_4}{\Delta}\left[\alpha_2\int_{\nu_2}^{\omega_2}\left(\int_0^{\sigma}\frac{(\sigma-\theta)^{\xi-2}}{\Gamma(\xi-1)}\mathfrak{M}_{h_1}d\theta\right)d\sigma\right.\\
&\qquad\left.+\beta_2\sum_{j=1}^{k-2}\upsilon_j\int_0^{\varsigma_j}\frac{(\varsigma_j-\sigma)^{\xi-2}}{\Gamma(\xi-1)}\mathfrak{M}_{h_1}d\sigma+\int_0^1\frac{(1-\sigma)^{\zeta-2}}{\Gamma(\zeta-1)}\mathfrak{M}_{h_2}d\sigma\right]\\
&\quad\le\frac{\mathfrak{M}_{h_1}}{\Gamma(\xi+1)}\Bigg[[(z_2-z_1)^{\xi}+(z_2^{\xi}-z_1^{\xi})]+\Big[\frac{(z_2-z_1)\vartheta_2}{\lambda_1}+\frac{(z_2^{n-1}-z_1^{n-1})\sigma_2}{\Delta}\Big](\rho_2(\omega_1^{\xi}-\nu_1^{\xi}))\\
&\qquad+\xi\Big[\frac{(z_2-z_1)\vartheta_3}{\lambda_1}+\frac{(z_2^{n-1}-z_1^{n-1})\sigma_3}{\Delta}\Big]+\Big[\frac{(z_2-z_1)\vartheta_4}{\lambda_1}+\frac{(z_2^{n-1}-z_1^{n-1})\sigma_4}{\Delta}\Big]\\
&\qquad\times\Big(\alpha_2(\omega_2^{\xi}-\nu_2^{\xi})+\xi\beta_2\sum_{j=1}^{k-2}\upsilon_j\varsigma_j^{\xi-1}\Big)\Bigg]\\
&\qquad+\frac{\mathfrak{M}_{h_2}}{\Gamma(\zeta+1)}\Bigg[\Big[\frac{(z_2-z_1)\vartheta_1}{\lambda_1}+\frac{(z_2^{n-1}-z_1^{n-1})\sigma_1}{\Delta}\Big](\rho_1(\omega_1^{\zeta}-\nu_1^{\zeta}))\\
&\qquad+\zeta\Big[\frac{(z_2-z_1)\vartheta_4}{\lambda_1}+\frac{(z_2^{n-1}-z_1^{n-1})\sigma_4}{\Delta}\Big]+\Big[\frac{(z_2-z_1)\vartheta_3}{\lambda_1}+\frac{(z_2^{n-1}-z_1^{n-1})\sigma_3}{\Delta}\Big]\\
&\qquad\times\Big(\alpha_1(\omega_2^{\zeta}-\nu_2^{\zeta})+\zeta\beta_1\sum_{j=1}^{k-2}\upsilon_j\varsigma_j^{\zeta-1}\Big)\Bigg],
\end{aligned}
$$

and

$$
\begin{aligned}
&|\mathfrak{T}_1'(p,q)(z_2)-\mathfrak{T}_1'(p,q)(z_1)|\\
&\qquad\le\frac{\mathfrak{M}_{h_1}}{\Gamma(\xi)}\Bigg[[(z_2-z_1)^{\xi-1}+(z_2^{\xi-1}-z_1^{\xi-1})]\\
&\qquad\quad+\Big[\frac{(n-1)(z_2^{n-2}-z_1^{n-2})\sigma_2}{\Delta}\Big]\frac{(\rho_2(\omega_1^{\xi}-\nu_1^{\xi}))}{\xi}\\
&\qquad\quad+\Big[\frac{(n-1)(z_2^{n-2}-z_1^{n-2})\sigma_3}{\Delta}\Big]+\Big[\frac{(n-1)(z_2^{n-2}-z_1^{n-2})\sigma_4}{\Delta}\Big]
\end{aligned}
$$

$$\times\Big(\frac{\alpha_2(\omega_2^{\xi}-\nu_2^{\xi})}{\xi}+\beta_2\sum_{j=1}^{k-2}\upsilon_j\varsigma_j^{\xi-1}\Big)\Bigg]$$
$$+\frac{\mathfrak{M}_{h_2}}{\Gamma(\zeta+1)}\Bigg[\Big[\frac{(n-1)(z_2^{n-2}-z_1^{n-2})\sigma_1}{\Delta}\Big](\rho_1(\omega_1^{\zeta}-\nu_1^{\zeta}))$$
$$+\zeta\Big[\frac{(n-1)(z_2^{n-2}-z_1^{n-2})\sigma_4}{\Delta}\Big]+\Big[\frac{(n-1)(z_2^{n-2}-z_1^{n-2})\sigma_3}{\Delta}\Big]$$
$$\times\Big(\alpha_1(\omega_2^{\zeta}-\nu_2^{\zeta})+\zeta\beta_1\sum_{j=1}^{k-2}\upsilon_j\varsigma_j^{\zeta-1}\Big)\Bigg].$$

Hence
$|{}^c\mathfrak{D}^{\gamma}\mathfrak{T}_1(p,q)(z_2)-{}^c\mathfrak{D}^{\gamma}\mathfrak{T}_1(p,q)(z_1)|$

$$\le \int_0^z\frac{(z-\sigma)^{-\gamma}}{\Gamma(1-\gamma)}|\mathfrak{T}'(p,q)(z_2)-\mathfrak{T}'(p,q)(z_1)|d\sigma$$
$$\le \frac{1}{\Gamma(2-\gamma)}\Bigg\{\frac{\mathfrak{M}_{h_1}}{\Gamma(\xi)}\Bigg[[(z_2-z_1)^{\xi-1}+(z_2^{\xi-1}-z_1^{\xi-1})]$$
$$+\Big[\frac{(n-1)(z_2^{n-2}-z_1^{n-2})\sigma_2}{\Delta}\Big]\frac{(\rho_2(\omega_1^{\xi}-\nu_1^{\xi}))}{\xi}$$
$$+\Big[\frac{(n-1)(z_2^{n-2}-z_1^{n-2})\sigma_3}{\Delta}\Big]+\Big[\frac{(n-1)(z_2^{n-2}-z_1^{n-2})\sigma_4}{\Delta}\Big]$$
$$\times\Big(\frac{\alpha_2(\omega_2^{\xi}-\nu_2^{\xi})}{\xi}+\beta_2\sum_{j=1}^{k-2}\upsilon_j\varsigma_j^{\xi-1}\Big)\Bigg]$$
$$+\frac{\mathfrak{M}_{h_2}}{\Gamma(\zeta+1)}\Bigg[\Big[\frac{(n-1)(z_2^{n-2}-z_1^{n-2})\sigma_1}{\Delta}\Big](\rho_1(\omega_1^{\zeta}-\nu_1^{\zeta}))$$
$$+\zeta\Big[\frac{(n-1)(z_2^{n-2}-z_1^{n-2})\sigma_4}{\Delta}\Big]+\Big[\frac{(n-1)(z_2^{n-2}-z_1^{n-2})\sigma_3}{\Delta}\Big]$$
$$\times\Big(\alpha_1(\omega_2^{\zeta}-\nu_2^{\zeta})+\zeta\beta_1\sum_{j=1}^{k-2}\upsilon_j\varsigma_j^{\zeta-1}\Big)\Bigg]\Bigg\}.$$

In consequence, $\|\mathfrak{T}_1(p,q)-\mathfrak{T}_1(p,q)\|_{\mathfrak{P}}\to 0$ independent of p and q as $z_2\to z_1$. In accordance with the above we obtain
$|\mathfrak{T}_2(p,q)(z_2)-\mathfrak{T}_2(p,q)(z_1)|$

$$\le \frac{\mathfrak{M}_{h_2}}{\Gamma(\zeta+1)}\Bigg[[(z_2-z_1)^{\zeta}+(z_2^{\zeta}-z_1^{\zeta})]$$
$$+\Big[\frac{(z_2-z_1)\vartheta_5}{\lambda_1}+\frac{(z_2^{n-1}-z_1^{n-1})\sigma_5}{\Delta}\Big](\rho_1(\omega_1^{\zeta}-\nu_1^{\zeta}))$$
$$+\zeta\Big[\frac{(z_2-z_1)\vartheta_8}{\lambda_1}+\frac{(z_2^{n-1}-z_1^{n-1})\sigma_8}{\Delta}\Big]$$

$$+\Big[\frac{(z_2-z_1)\vartheta_7}{\lambda_1}+\frac{(z_2^{n-1}-z_1^{n-1})\sigma_7}{\Delta}\Big][\alpha_1(\omega_2^{\zeta}-\nu_2^{\zeta})+\zeta\beta_1\sum_{j=1}^{k-2}\upsilon_j\varsigma_j^{\zeta-1}]\Bigg]$$
$$+\frac{\mathfrak{M}_{h_1}}{\Gamma(\xi+1)}\Bigg[\Big[\frac{(z_2-z_1)\vartheta_6}{\lambda_1}+\frac{(z_2^{n-1}-z_1^{n-1})\sigma_6}{\Delta}\Big](\rho_2(\omega_1^{\xi}-\nu_1^{\xi}))$$
$$+\xi\Big[\frac{(z_2-z_1)\vartheta_7}{\lambda_1}+\frac{(z_2^{n-1}-z_1^{n-1})\sigma_7}{\Delta}\Big]$$
$$+\Big[\frac{(z_2-z_1)\vartheta_8}{\lambda_1}+\frac{(z_2^{n-1}-z_1^{n-1})\sigma_8}{\Delta}\Big][\alpha_2(\omega_2^{\xi}-\nu_2^{\xi})+\xi\beta_2\sum_{j=1}^{k-2}\upsilon_j\varsigma_j^{\xi-1}]\Bigg],$$

$$|\mathfrak{T}_2'(p,q)(z_2)-\mathfrak{T}_2'(p,q)(z_1)|$$
$$\leq \frac{\mathfrak{M}_{h_2}}{\Gamma(\zeta)}\Bigg[[(z_2-z_1)^{\zeta-1}+(z_2^{\zeta-1}-z_1^{\zeta-1})]$$
$$+\Big[\frac{(n-1)(z_2^{n-2}-z_1^{n-2})\sigma_5}{\Delta}\Big]\frac{(\rho_1(\omega_1^{\zeta}-\nu_1^{\zeta}))}{\zeta}$$
$$+\Big[\frac{(n-1)(z_2^{n-2}-z_1^{n-2})\sigma_8}{\Delta}\Big]+\Big[\frac{(n-1)(z_2^{n-2}-z_1^{n-2})\sigma_7}{\Delta}\Big]$$
$$\times\Big[\frac{\alpha_1(\omega_2^{\zeta}-\nu_2^{\zeta})}{\zeta}+\beta_1\sum_{j=1}^{k-2}\upsilon_j\varsigma_j^{\zeta-1}\Big]\Bigg]$$
$$+\frac{\mathfrak{M}_{h_1}}{\Gamma(\xi+1)}\Bigg[\Big[\frac{(n-1)(z_2^{n-2}-z_1^{n-2})\sigma_6}{\Delta}\Big](\rho_2(\omega_1^{\xi}-\nu_1^{\xi}))$$
$$+\xi\Big[\frac{(n-1)(z_2^{n-2}-z_1^{n-2})\sigma_7}{\Delta}\Big]+\Big[\frac{(n-1)(z_2^{n-2}-z_1^{n-2})\sigma_8}{\Delta}\Big]$$
$$\times[\alpha_2(\omega_2^{\xi}-\nu_2^{\xi})+\xi\beta_2\sum_{j=1}^{k-2}\upsilon_j\varsigma_j^{\xi-1}]\Bigg].$$

and

$$|{}^c\mathfrak{D}^{\delta}\mathfrak{T}_2(p,q)(z_2)-{}^c\mathfrak{D}^{\delta}\mathfrak{T}_2(p,q)(z_1)|$$
$$\leq \frac{1}{\Gamma(2-\delta)}\Bigg\{\frac{\mathfrak{M}_{h_2}}{\Gamma(\zeta)}\Bigg[[(z_2-z_1)^{\zeta-1}+(z_2^{\zeta-1}-z_1^{\zeta-1})]$$
$$+\Big[\frac{(n-1)(z_2^{n-2}-z_1^{n-2})\sigma_5}{\Delta}\Big]\frac{(\rho_1(\omega_1^{\zeta}-\nu_1^{\zeta}))}{\zeta}$$
$$+\Big[\frac{(n-1)(z_2^{n-2}-z_1^{n-2})\sigma_8}{\Delta}\Big]+\Big[\frac{(n-1)(z_2^{n-2}-z_1^{n-2})\sigma_7}{\Delta}\Big]$$
$$\times\Big[\frac{\alpha_1(\omega_2^{\zeta}-\nu_2^{\zeta})}{\zeta}+\beta_1\sum_{j=1}^{k-2}\upsilon_j\varsigma_j^{\zeta-1}\Big]\Bigg]$$
$$+\frac{\mathfrak{M}_{h_1}}{\Gamma(\xi+1)}\Bigg[\Big[\frac{(n-1)(z_2^{n-2}-z_1^{n-2})\sigma_6}{\Delta}\Big](\rho_2(\omega_1^{\xi}-\nu_1^{\xi}))$$

$$+\xi\Big[\frac{(n-1)(z_2^{n-2}-z_1^{n-2})\sigma_7}{\Delta}\Big]+\Big[\frac{(n-1)(z_2^{n-2}-z_1^{n-2})\sigma_8}{\Delta}\Big]$$
$$\times[\alpha_2(\omega_2^{\xi}-\nu_2^{\xi})+\xi\beta_2\sum_{j=1}^{k-2}\upsilon_j\varsigma_j^{\xi-1}]\Big]\Big\},$$

which imply that $\|\mathfrak{T}_2(p,q)-\mathfrak{T}_2(p,q)\|_{\mathfrak{Q}}\to 0$ independent of p and q as $z_2\to z_1$. Therefore, the operator $\mathfrak{T}(p,q)$ is equicontinuous and consequently, by Arzela-Ascoli theorem, it is completely continuous. Next, we show that the set $\mathfrak{W}=\{(p,q)\in\mathfrak{P}\times\mathfrak{Q}|(p,q)=\kappa\mathfrak{T}(p,q),0<\kappa<1\}$ is bounded. Let $(p,q)\in\mathfrak{W}$, then $(p,q)=\kappa\mathfrak{T}(p,q)$ and for any $z\in\mathfrak{J}$, we have

$$p(z) = \kappa\ \mathfrak{T}_1(p,q)(z), \qquad q(z) = \kappa\ \mathfrak{T}_2(p,q)(z).$$

Thus,

$$\begin{aligned}|p(z)| &\le \Omega_1(\tau_0+\tau_1\|p\|_{\mathfrak{P}}+max\{\tau_2,\tau_3\}\|q\|_{\mathfrak{Q}})+\mu_2\\ &\quad+\hat{\Omega}_1(\varepsilon_0+max\{\varepsilon_1,\varepsilon_2\}\|p\|_{\mathfrak{P}}+\varepsilon_3\|q\|_{\mathfrak{Q}}),\end{aligned}\tag{22}$$

$$\begin{aligned}|p^{'}(z)| &\le \Omega_2(\tau_0+\tau_1\|p\|_{\mathfrak{P}}+max\{\tau_2,\tau_3\}\|q\|_{\mathfrak{Q}})\\ &\quad+\hat{\Omega}_2(\varepsilon_0+max\{\varepsilon_1,\varepsilon_2\}\|p\|_{\mathfrak{P}}+\varepsilon_3\|q\|_{\mathfrak{Q}}),\end{aligned}$$

and

$$\begin{aligned}|{}^c\mathfrak{D}^{\gamma}p(z)| &\le \frac{1}{\Gamma(2-\gamma)}\Big[\Omega_2(\tau_0+\tau_1\|p\|_{\mathfrak{P}}+max\{\tau_2,\tau_3\}\|q\|_{\mathfrak{Q}})\\ &\quad+\hat{\Omega}_2(\varepsilon_0+max\{\varepsilon_1,\varepsilon_2\}\|p\|_{\mathfrak{P}}+\varepsilon_3\|q\|_{\mathfrak{Q}})\Big].\end{aligned}\tag{23}$$

Using inequalities (22)-(23), we get

$$\begin{aligned}\|p\|_{\mathfrak{P}} &\le \Omega_1(\tau_0+\tau_1\|p\|_{\mathfrak{P}}+max\{\tau_2,\tau_3\}\|q\|_{\mathfrak{Q}})+\mu_2\\ &\quad+\hat{\Omega}_1(\varepsilon_0+max\{\varepsilon_1,\varepsilon_2\}\|p\|_{\mathfrak{P}}+\varepsilon_3\|q\|_{\mathfrak{Q}})\\ &\quad+\frac{1}{\Gamma(2-\gamma)}\Big[\Omega_2(\tau_0+\tau_1\|p\|_{\mathfrak{P}}+max\{\tau_2,\tau_3\}\|q\|_{\mathfrak{Q}})\\ &\quad+\hat{\Omega}_2(\varepsilon_0+max\{\varepsilon_1,\varepsilon_2\}\|p\|_{\mathfrak{P}}+\varepsilon_3\|q\|_{\mathfrak{Q}})\Big].\end{aligned}\tag{24}$$

Identical to the above we can have

$$\begin{aligned}\|q\|_{\mathfrak{Q}} &\le \Omega_3(\varepsilon_0+max\{\varepsilon_1,\varepsilon_2\}\|p\|_{\mathfrak{P}}+\varepsilon_3\|q\|_{\mathfrak{Q}})+\mu_1\\ &\quad+\hat{\Omega}_3(\tau_0+\tau_1\|p\|_{\mathfrak{P}}+max\{\tau_2,\tau_3\}\|q\|_{\mathfrak{Q}})\\ &\quad+\frac{1}{\Gamma(2-\delta)}\Big[\Omega_4(\varepsilon_0+max\{\varepsilon_1,\varepsilon_2\}\|p\|_{\mathfrak{P}}+\varepsilon_3\|q\|_{\mathfrak{Q}})\\ &\quad+\hat{\Omega}_4(\tau_0+\tau_1\|p\|_{\mathfrak{P}}+max\{\tau_2,\tau_3\}\|q\|_{\mathfrak{Q}})\Big].\end{aligned}\tag{25}$$

The above inequalities together with the notations (24)-(25) yields, $\|p\|_{\mathfrak{P}}+\|q\|_{\mathfrak{Q}}\le\mathfrak{V}_1+max\{\mathfrak{V}_2,\mathfrak{V}_3\}\|(p,q)\|_{\mathfrak{P}\times\mathfrak{Q}}$, which leads to $\|(p,q)\|_{\mathfrak{P}\times\mathfrak{Q}}\le\frac{\mathfrak{V}_1}{1-\hat{\mathfrak{V}}}$. This concludes that the set is $\mathfrak{W}$ is bounded. Thus, by Theorem 3.1, the operator $\mathfrak{T}$ has at least one fixed point, which implies that the problem (1) and (2) has at least one solution on $\mathfrak{J}$. □

Example 3.3 Consider the fractional differential equations given by

$$(26)\quad \begin{cases} {}^{c}\mathfrak{D}^{\frac{5}{2}}p(z)=\dfrac{\tan^{-1}z}{3\sqrt{z+16}}+\dfrac{\sin p(z)}{175}+\dfrac{q(z)|p(z)|}{350(1+|p(z)|)}+\dfrac{{}^{c}\mathfrak{D}^{\frac{4}{7}}q(z)}{\sqrt{z^4+400}},\\ {}^{c}\mathfrak{D}^{\frac{7}{3}}q(z)=\dfrac{\sin z}{5\sqrt{z^2+36}}+\dfrac{\cos p(z)}{270(1+q(z)))}+\dfrac{\tan^{-1}q(z)}{9\sqrt{z+324}}+\dfrac{{}^{c}\mathfrak{D}^{\frac{5}{9}}p(z)}{30\sqrt{z^2+1600}}, \end{cases}$$

subject to the boundary conditions

$$(27)\quad \begin{cases} p(0)=\dfrac{q(z)}{10},p'(0)=\rho_1\displaystyle\int_{\frac{1}{7}}^{\frac{1}{5}}q'(\sigma)d\sigma,p'(1)=\alpha_1\int_{\frac{1}{4}}^{\frac{1}{3}}q'(\sigma)d\sigma+\beta_1\sum_{j=1}^{4}\upsilon_j q'(\varsigma_j),\\ q(0)=\dfrac{p(z)}{11},q'(0)=\rho_2\displaystyle\int_{\frac{1}{7}}^{\frac{1}{5}}p'(\sigma)d\sigma,q'(1)=\alpha_2\int_{\frac{1}{4}}^{\frac{1}{3}}p'(\sigma)d\sigma+\beta_2\sum_{j=1}^{4}\upsilon_j p'(\varsigma_j). \end{cases}$$

Clearly,

$$\begin{aligned} |h_1(z,p(z),q(z),{}^{c}\mathfrak{D}^{\frac{4}{7}}q(z))| &\le \frac{1}{12}+\frac{1}{175}\|p\|+\frac{1}{350}\|q\|+\frac{1}{20}\|{}^{c}\mathfrak{D}^{\frac{4}{7}}q\|,\\ |h_2(z,p(z),{}^{c}\mathfrak{D}^{\frac{5}{8}}p(z),q(z))| &\le \frac{1}{30}+\frac{1}{270}\|p\|+\frac{1}{162}\|{}^{c}\mathfrak{D}^{\frac{5}{8}}p\|+\frac{1}{1200}\|q\|. \end{aligned}$$

Here, $\xi=\frac{5}{2}$, $\zeta=\frac{7}{3}$, $\delta=\frac{4}{7}$, $\gamma=\frac{5}{8}$, $\rho_1=\frac{1}{3}$, $\rho_2=\frac{1}{5}$, $\alpha_1=\beta_1=\frac{1}{2}$, $\alpha_2=\beta_2=\frac{3}{4}$, $\upsilon_1=\frac{2}{7}$, $\upsilon_2=\frac{4}{11}$, $\upsilon_3=\frac{3}{8}$, $\upsilon_4=\frac{5}{12}$, $\varsigma_1=\frac{3}{10}$, $\varsigma_2=\frac{4}{9}$, $\varsigma_3=\frac{7}{13}$, $\varsigma_4=\frac{8}{11}$, $\tau_0=\frac{1}{12}$, $\tau_1=\frac{1}{175}$, $\tau_2=\frac{1}{350}$, $\tau_3=\frac{1}{20}$, $\varepsilon_0=\frac{1}{30}$, $\varepsilon_1=\frac{1}{270}$, $\varepsilon_2=\frac{1}{162}$, $\varepsilon_3=\frac{1}{1200}$, $\mu_1=\frac{1}{10}$, $\mu_2=\frac{1}{11}$, $\nu_1=\frac{1}{7}$,$\omega_1=\frac{1}{5}$,$\nu_2=\frac{1}{3}$, $\omega_2=\frac{1}{2}$. With the given data, we find that $\varpi_1=0.01904$, $\varpi_2=0.00653$, $\varpi_3=0.01142$, $\varpi_4=0.00391$, $\eta_1=0.80384$, $\eta_2=0.82172$, $\eta_3=1.20576$, $\eta_4=1.23259$, $\lambda_1=0.99978$, $\lambda_2=0.03075$, $\Delta_1=1.00897$, $\Delta_2=1.1788$, $\Delta_3=0.78598$, $\Delta_4=1.00897$, $\Delta=-0.091509$, $\sigma_1=0.03075$, $\sigma_2=-0.02358$, $\sigma_3=0.06124$, $\sigma_4=0.02506$, $\sigma_5=0.03590$, $\sigma_6=-0.03034$, $\sigma_7=-0.03777$, $\sigma_8=-0.06139$,$\vartheta_1=1.00254$, $\vartheta_2=0.01690$, $\vartheta_3=0.00272$, $\vartheta_4=0.00441$, $\vartheta_5=0.01014$, $\vartheta_6=1.00099$, $\vartheta_7=0.00265$, $\vartheta_8=0.00112$, $\Phi_1=0.66664$, $\Phi_2=0.27462$, $\Phi_3=0.67202$, $\Phi_4=0.27829$, $\Phi_5=0.40247$, $\Phi_6=0.66962$, $\Phi_7=0.41547$, $\Phi_8=0.67201$, $\Phi_1'=0.67221$, $\Phi_2'=0.51543$, $\Phi_3'=1.33861$, $\Phi_4'=0.54779$, $\Phi_5'=0.78466$, $\Phi_6'=0.66315$, $\Phi_7'=0.82563$, $\Phi_8'=1.34179$, $\Omega_1=0.90180$, $\hat{\Omega}_1=0.42370$, $\Omega_2=1.94694$, $\hat{\Omega}_2=0.83542$, $\Omega_3=1.0418$, $\hat{\Omega}_3=0.54284$, $\Omega_4=2.20017$, $\hat{\Omega}_4=1.08051$, we find that $\mathfrak{V}_2=0.14882$, $\mathfrak{V}_3=0.34679$, $\hat{\mathfrak{V}}=max\{\mathfrak{V}_2,\mathfrak{V}_3\}\cong 0.34679<1$. Thus, the assumptions of Theorem 3.2 holds and the problem (26) and (27) has at least one solution on $\mathfrak{J}$.

Next, we shall roll the ball towards the uniqueness of solutions using Banach fixed point theorem for problem (1) and (2). To run the interference for the proof, we introduce the notations:

$$(28)\quad \Theta = \Theta_1+\frac{\hat{\Theta}_1}{\Gamma(2-\gamma)},\quad \hat{\Theta} = \Theta_2+\frac{\hat{\Theta}_2}{\Gamma(2-\delta)},$$

$$(29)\quad \mathfrak{S} = \mathfrak{S}_1+\frac{\hat{\mathfrak{S}}_1}{\Gamma(2-\gamma)},\quad \hat{\mathfrak{S}} = \mathfrak{S}_2+\frac{\hat{\mathfrak{S}}_2}{\Gamma(2-\delta)},$$

$$(30)\quad \Theta_1 = \Omega_1\mathfrak{G}_1+\hat{\Omega}_1\mathfrak{G}_2+\varrho_1,\qquad \mathfrak{S}_1=\Omega_1\mathfrak{L}_1+\hat{\Omega}_1\mathfrak{L}_2,$$

$$(31)\quad \hat{\Theta}_1 = \Omega_2\mathfrak{G}_1+\hat{\Omega}_2\mathfrak{G}_2,\qquad \hat{\mathfrak{S}}_1=\Omega_2\mathfrak{L}_1+\hat{\Omega}_2\mathfrak{L}_2,$$

$$\begin{aligned}
&(32) \qquad \Theta_2 = \Omega_3\mathfrak{G}_2 + \hat{\Omega}_3\mathfrak{G}_1 + \varrho_2, \quad \mathfrak{S}_2 = \Omega_3\mathfrak{L}_2 + \hat{\Omega}_3\mathfrak{L}_1,\\
&(33) \qquad \hat{\Theta}_2 = \Omega_4\mathfrak{G}_2 + \hat{\Omega}_4\mathfrak{G}_1, \quad \hat{\mathfrak{S}}_2 = \Omega_4\mathfrak{L}_2 + \hat{\Omega}_4\mathfrak{L}_1,\\
&\qquad \mathfrak{L}_1 = \sup_{z\in\mathfrak{J}}|h_1(z,0,0,0)| < \infty, \quad \mathfrak{L}_2 = \sup_{z\in\mathfrak{J}}|h_2(z,0,0,0)| < \infty.
\end{aligned}$$

Theorem 3.4. *Let us speculate that* $(\mathfrak{H}_3)$ *and* $(\mathfrak{H}_4)$ *holds. In addition that*

$$\Theta + \hat{\Theta} < 1, \tag{34}$$

where Θ, $\hat{\Theta}$ *are defined by (28). Then* $\exists$ *a unique solution for problem (1)-(2) on* $\mathfrak{J}$.

Proof. Let us define $r > \frac{\mathfrak{S}+\hat{\mathfrak{S}}}{1-(\Theta+\hat{\Theta})}$. where $\Theta, \hat{\Theta}$ and $\mathfrak{S}, \hat{\mathfrak{S}}$ are respectively given by (28) and (29), and show that $\mathfrak{T}\mathfrak{B}_r \subset \mathfrak{B}_r$, where the operator $\mathfrak{T}$ is given by (13) and $\mathfrak{B}_r \geq \{(p,q) \in \mathfrak{P}\times\mathfrak{Q} : \|(p,q)\|_{\mathfrak{P}\times\mathfrak{Q}} \leq r\}$. In view of the assumption $(\mathfrak{H}_3)$ and $(\mathfrak{H}_4)$ with (30) and (32), for $(p,q) \in \mathfrak{B}_r, z \in \mathfrak{J}$, we have

$$\begin{aligned}
|h_1(z,p(z),q(z),{}^c\mathfrak{D}^\delta q(z))| &\leq \mathfrak{G}_1(|p(z)| + |q(z)| + |{}^c\mathfrak{D}^\delta q(z)|) + \mathfrak{L}_1\\
&\leq \mathfrak{G}_1(\|p(z)\|_{\mathfrak{P}} + \|q(z)\|_{\mathfrak{Q}}) + \mathfrak{L}_1 \leq \mathfrak{G}_1 r + \mathfrak{L}_1,
\end{aligned}$$

$$|\varphi_1(q)| \leq \varrho_1\|q\| \leq \varrho_1\|q\|_{\mathfrak{Q}} \leq \varrho_1 r, \quad |\varphi_2(p)| \leq \varrho_2\|p\|_{\mathfrak{P}} \leq \varrho_2 r,$$

$$|h_2(z,p(z),{}^c\mathfrak{D}^\gamma p(z),q(z))| \leq \mathfrak{G}_2(\|p(z)\|_{\mathfrak{P}} + \|q(z)\|_{\mathfrak{Q}}) + \mathfrak{L}_2 \leq \mathfrak{G}_2 r + \mathfrak{L}_2.$$

This guides to

$$\begin{aligned}
|\mathfrak{T}_1(p,q)(z)| \leq{}& \int_0^z \frac{(z-\sigma)^{\xi-1}}{\Gamma(\xi)}(\mathfrak{G}_1 r + \mathfrak{L}_1)d\sigma + (\varrho_1 r)\\
&+\hat{\Phi}_1(z)\left[\rho_1\int_{\nu_1}^{\omega_1}\left(\int_0^\sigma \frac{(\sigma-\theta)^{\zeta-2}}{\Gamma(\zeta-1)}(\mathfrak{G}_2 r + \mathfrak{L}_2)d\theta\right)d\sigma\right]\\
&+\hat{\Phi}_2(z)\left[\rho_2\int_{\nu_1}^{\omega_1}\left(\int_0^\sigma \frac{(\sigma-\theta)^{\xi-2}}{\Gamma(\xi-1)}(\mathfrak{G}_1 r + \mathfrak{L}_1)d\theta\right)d\sigma\right]\\
&+\hat{\Phi}_3(z)\left[\alpha_1\int_{\nu_2}^{\omega_2}\left(\int_0^\sigma \frac{(\sigma-\theta)^{\zeta-2}}{\Gamma(\zeta-1)}(\mathfrak{G}_2 r + \mathfrak{L}_2)d\theta\right)d\sigma\right.\\
&+\beta_1\sum_{j=1}^{k-2}\upsilon_j\int_0^{\varsigma_j}\frac{(\varsigma_j-\sigma)^{\zeta-2}}{\Gamma(\zeta-1)}(\mathfrak{G}_2 r + \mathfrak{L}_2)d\sigma\\
&\left.+\int_0^1 \frac{(1-\sigma)^{\xi-2}}{\Gamma(\xi-1)}(\mathfrak{G}_1 r + \mathfrak{L}_1)d\sigma\right]\\
&+\hat{\Phi}_4(z)\left[\alpha_2\int_{\nu_2}^{\omega_2}\left(\int_0^\sigma \frac{(\sigma-\theta)^{\xi-2}}{\Gamma(\xi-1)}(\mathfrak{G}_1 r + \mathfrak{L}_1)d\theta\right)d\sigma\right.\\
&+\beta_2\sum_{j=1}^{k-2}\upsilon_j\int_0^{\varsigma_j}\frac{(\varsigma_j-\sigma)^{\xi-2}}{\Gamma(\xi-1)}(\mathfrak{G}_1 r + \mathfrak{L}_1)d\sigma\\
&\left.+\int_0^1 \frac{(1-\sigma)^{\zeta-2}}{\Gamma(\zeta-1)}(\mathfrak{G}_2 r + \mathfrak{L}_2)d\sigma\right] \leq \Theta_1 r + \mathfrak{S}_1,
\end{aligned}$$

and

$$
\begin{aligned}
|\mathfrak{T}_1'(p,q)(z)| \leq & \int_0^z \frac{(z-\sigma)^{\xi-2}}{\Gamma(\xi-1)}(\mathfrak{G}_1 r+\mathfrak{L}_1)d\sigma \\
& +\hat{\Phi}_1'(z)\left[\rho_1\int_{\nu_1}^{\omega_1}\left(\int_0^{\sigma}\frac{(\sigma-\theta)^{\zeta-2}}{\Gamma(\zeta-1)}(\mathfrak{G}_2 r+\mathfrak{L}_2)d\theta\right)d\sigma\right] \\
& +\hat{\Phi}_2'(z)\left[\rho_2\int_{\nu_1}^{\omega_1}\left(\int_0^{\sigma}\frac{(\sigma-\theta)^{\xi-2}}{\Gamma(\xi-1)}(\mathfrak{G}_1 r+\mathfrak{L}_1)d\theta\right)d\sigma\right] \\
& +\hat{\Phi}_3'(z)\left[\alpha_1\int_{\nu_2}^{\omega_2}\left(\int_0^{\sigma}\frac{(\sigma-\theta)^{\zeta-2}}{\Gamma(\zeta-1)}(\mathfrak{G}_2 r+\mathfrak{L}_2)d\theta\right)d\sigma\right. \\
& +\beta_1\sum_{j=1}^{k-2}\upsilon_j\int_0^{\varsigma_j}\frac{(\varsigma_j-\sigma)^{\zeta-2}}{\Gamma(\zeta-1)}(\mathfrak{G}_2 r+\mathfrak{L}_2)d\sigma \\
& \left.+\int_0^1\frac{(1-\sigma)^{\xi-2}}{\Gamma(\xi-1)}(\mathfrak{G}_1 r+\mathfrak{L}_1)d\sigma\right] \\
& +\hat{\Phi}_4'(z)\left[\alpha_2\int_{\nu_2}^{\omega_2}\left(\int_0^{\sigma}\frac{(\sigma-\theta)^{\xi-2}}{\Gamma(\xi-1)}(\mathfrak{G}_1 r+\mathfrak{L}_1)d\theta\right)d\sigma\right. \\
& +\beta_2\sum_{j=1}^{k-2}\upsilon_j\int_0^{\varsigma_j}\frac{(\varsigma_j-\sigma)^{\xi-2}}{\Gamma(\xi-1)}(\mathfrak{G}_1 r+\mathfrak{L}_1)d\sigma \\
& \left.+\int_0^1\frac{(1-\sigma)^{\zeta-2}}{\Gamma(\zeta-1)}(\mathfrak{G}_2 r+\mathfrak{L}_2)d\sigma\right] \leq \hat{\Theta}_1 r+\hat{\mathfrak{S}}_1.
\end{aligned}
$$

The above inequality infers that,

$$
|{}^c\mathfrak{D}^{\gamma}\mathfrak{T}_1(p,q)(z)| \leq \int_0^z\frac{(z-\sigma)^{-\gamma}}{\Gamma(1-\gamma)}|\mathfrak{T}_1'(p,q)(\sigma)|d\sigma \leq \frac{1}{\Gamma(2-\gamma)}\left[\hat{\Theta}_1 r+\hat{\mathfrak{S}}_1\right].
$$

Thus

$$
\|\mathfrak{T}_1(p,q)\|_{\mathfrak{P}} \leq \left(\Theta_1+\frac{\hat{\Theta}_1}{\Gamma(2-\gamma)}\right)r+\left(\mathfrak{S}_1+\frac{\hat{\mathfrak{S}}_1}{\Gamma(2-\gamma)}\right) \leq \Theta r+\mathfrak{G}. \tag{35}
$$

Equivalently, we obtain

$$
|\mathfrak{T}_2(p,q)(z)| \leq \Theta_2 r+\mathfrak{S}_2.
$$

$$
\begin{aligned}
|\mathfrak{T}_2'(p,q)(z)| &\leq \hat{\Theta}_2 r+\hat{\mathfrak{S}}_2 \\
|{}^c\mathfrak{D}^{\delta}\mathfrak{T}_2(p,q)(z)| &\leq \int_0^z\frac{(z-\sigma)^{-\delta}}{\Gamma(1-\delta)}|\mathfrak{T}_2'(p,q)(\sigma)|d\sigma \leq \frac{1}{\Gamma(2-\delta)}\left[\hat{\Theta}_2 r+\hat{\mathfrak{S}}_2\right],
\end{aligned}
$$

which yields

$$
\|\mathfrak{T}_2(p,q)\|_{\mathfrak{Q}} \leq \left(\Theta_2+\frac{\hat{\Theta}_2}{\Gamma(2-\delta)}\right)r+\left(\mathfrak{S}_2+\frac{\hat{\mathfrak{S}}_2}{\Gamma(2-\delta)}\right) \leq \hat{\Theta} r+\hat{\mathfrak{S}}. \tag{36}
$$

Thus, it follows from (35) and (36) that $\|\mathfrak{T}(p,q)\|_{\mathfrak{P}\times\mathfrak{Q}} \leq r$, and consequently, $\mathfrak{T}\mathfrak{B}_r \subset \mathfrak{B}_r$. In view of $(\mathfrak{H}_3)$ and $(\mathfrak{H}_4)$, it follows that

$$
|\varphi_1(q_1)-\varphi_1(q_2)| \leq \varrho_1(\|q_1-q_2\|), \tag{37}
$$

$$(38) \qquad |\varphi_2(p_1) - \varphi_2(p_2)| \leq \varrho_2(\|p_1 - p_2\|).$$

Using (37)-(38), we obtain

$$|\mathfrak{T}_1(p_1, q_1)(z) - \mathfrak{T}_1(p_2, q_2)(z)| \leq \Theta_1(\|p_1 - p_2\|_{\mathfrak{P}} + \|q_1 - q_2\|_{\mathfrak{Q}}),$$

and

$$(39) \qquad |\mathfrak{T}_1'(p_1, q_1)(z) - \mathfrak{T}_1'(p_2, q_2)(z)| \leq \hat{\Theta}_1(\|p_1 - p_2\|_{\mathfrak{P}} + \|q_1 - q_2\|_{\mathfrak{Q}}).$$

By the generosity of (39), we have

$$|{}^c\mathfrak{D}^\gamma \mathfrak{T}_1(p_1, q_1)(z) - {}^c\mathfrak{D}^\gamma \mathfrak{T}_1(p_2, q_2)(z)| \leq \frac{1}{\Gamma(2-\gamma)}\Big[\hat{\Theta}_1(\|p_1 - p_2\|_{\mathfrak{P}} + \|q_1 - q_2\|_{\mathfrak{Q}})\Big].$$

This guides to

$$\|\mathfrak{T}_1(p_1, q_1) - \mathfrak{T}_1(p_2, q_2)\|_{\mathfrak{P}} \leq \Theta(\|p_1 - p_2\|_{\mathfrak{P}} + \|q_1 - q_2\|_{\mathfrak{Q}}).$$

Equivalently, we obtain

$$|\mathfrak{T}_2(p_1, q_1)(z) - \mathfrak{T}_2(p_2, q_2)(z)| \leq \Theta_2(\|p_1 - p_2\|_{\mathfrak{P}} + \|q_1 - q_2\|_{\mathfrak{Q}}),$$

and

$$|\mathfrak{T}_2'(p_1, q_1)(z) - \mathfrak{T}_2'(p_2, q_2)(z)| \leq \hat{\Theta}_2(\|p_1 - p_2\|_{\mathfrak{P}} + \|q_1 - q_2\|_{\mathfrak{Q}}).$$

The above inequality infers that,

$$|{}^c\mathfrak{D}^\delta \mathfrak{T}_2(p_1, q_1)(z) - {}^c\mathfrak{D}^\delta \mathfrak{T}_2(p_2, q_2)(z)| \leq \frac{1}{\Gamma(2-\delta)}\Big[\hat{\Theta}_2(\|p_1 - p_2\|_{\mathfrak{P}} + \|q_1 - q_2\|_{\mathfrak{Q}})\Big].$$

Equivalently, we obtain

$$\|\mathfrak{T}_2(p_1, q_1) - \mathfrak{T}_2(p_2, q_2)\|_{\mathfrak{Q}} \leq \hat{\Theta}(\|p_1 - p_2\|_{\mathfrak{P}} + \|q_1 - q_2\|_{\mathfrak{Q}}).$$

Thus, we obtain

$$(40) \qquad \|\mathfrak{T}(p_1, q_1) - \mathfrak{T}(p_2, q_2)\|_{\mathfrak{P}\times\mathfrak{Q}} \leq (\Theta + \hat{\Theta})(\|p_1 - p_2\|_{\mathfrak{P}} + \|q_1 - q_2\|_{\mathfrak{Q}}).$$

Since $\Theta + \hat{\Theta} \in (0, 1)$ by the given assumption (34), therefore $\mathfrak{T}$ is a contraction. Hence it follows by Banach fixed point theorem that the equation (1) and (2) has at most one solution. □

EXAMPLE 3.5 Consider the fractional differential equation given by

$$(41) \qquad \begin{cases} \mathfrak{D}^{\frac{8}{3}} p(z) = \dfrac{1}{4\sqrt{81+z}}\Big({}^c\mathfrak{D}^{\frac{3}{5}} q(z) + \sin q(z) + p(z)\Big) + e^{-z}, \\ \mathfrak{D}^{\frac{9}{4}} q(z) = \dfrac{1}{6\sqrt{144+z}}\Big(\tan^{-1}({}^c\mathfrak{D}^{\frac{4}{9}} p(z)) + \cos q(z) + \cos p(z)\Big) + \sin z, \end{cases}$$

subject to the boundary conditions (27). Using the given data, it is found that $\mathfrak{G}_1 = \frac{1}{36}$, $\mathfrak{G}_2 = \frac{1}{72}$, $\varrho_1 = \frac{1}{10}$, $\varrho_2 = \frac{1}{11}$, $\Omega_1 = 0.77234$, $\hat{\Omega}_1 = 0.45568$, $\Omega_2 = 1.7048$, $\hat{\Omega}_2 = 0.89818$, $\Omega_3 = 1.11525$, $\hat{\Omega}_3 = 0.46074$, $\Omega_4 = 2.32489$, $\hat{\Omega}_4 = 0.91707$, $\Theta_1 = 0.12778$, $\hat{\Theta}_1 = 0.05983$, $\Theta_2 = 0.11919$, $\hat{\Theta}_2 = 0.05776$, $\Theta = 0.195062$, $\hat{\Theta} = 0.184301$. With $\Theta + \hat{\Theta} \cong 0.379363 < 1$, the assumption of Theorem 3.4 holds and hence the problem (41) with the boundary conditions (27) has unique solution on $\mathfrak{J}$.

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